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Stability analysis of the discrete Landau–Ginsburg equation

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Abstract. The stability of time-independent solutions of a class of discrete nonlinear equations is investigated by extending a method developed earlier to study the stability of the static solutions of the continuous Landau–Ginsburg equation. A simple necessary condition for stability is found and it is shown that all nonlinear wave solutions are unstable while soliton and kink solutions may be stable. A further method is introduced which shows that the soliton solution is in fact unstable whilst the kink is marginally stable.

1. Introduction

In a recent paper the present author and others (Grundland, Infeld, Rowlands and Winternitz 1990: hereafter referred to as GIRW) carried-out a stability analysis of a class of solutions of the nonlinear Landau–Ginsburg equation. It was shown that a necessary condition for the stability of nonlinear wave solutions to long wavelength disturbances, was that $\partial\lambda/\partial a < 0$: here λ is the wavelength of the nonlinear wave and a is a measure of the amplitude of this wave. In particular, soliton and kink type solutions are stable. In this paper we extend this method of analysis to encompass the discrete analogue of the Landau–Ginsburg equation. We write this equation in the form

$$(1/\gamma)\frac{\partial M_n}{\partial t} = 2(M_{n+1} + M_{n-1} - 2M_n) - \dot{V}(M_n) \tag{1.1}$$

where we may consider M_n to be the order parameter at the nth point of a one-dimensional lattice. $V(M_n)$ is the local free energy at the same lattice point and the dot denotes differentiation with respect to the argument, whilst γ is a measure of the damping in the system. Such equations arise naturally in solid state physics where the lattice is the atomic lattice. If we replace the left-hand side of the above equation and write

$$a\frac{\partial^2 M_n}{\partial t^2} = 2(M_{n+1} + M_{n-1} - 2M_n) - \dot{V}(M_n) \tag{1.2}$$

we have an equation which describes molecular vibrations on an ordered chain such as a protein molecule (see for example Zhang 1990).

The usual continuous Landau–Ginsburg equation can be obtained from equation (1.1) in the limit as the variation of M_n with n is slow compared to the underlying lattice. Then we may write $M_n = M(n\Delta) = M(\xi)$, where Δ is the lattice spacing, and Taylor series expand $M_{n\pm1} = M(n\Delta \pm \Delta) = M(\xi \pm \Delta)$ to give the form discussed in GIRW

$$1/\gamma\frac{\partial M}{\partial t} = 2\frac{\partial^2 M}{\partial x^2} - \dot{V}(M(x)) \qquad (x = \xi/\Delta). \tag{1.3}$$

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In analogy with the treatment given in GIRW we first look for time-independent solutions of the above equations. In the continuous case one uses the powerful methods of phase plane analysis to show the existence, in general, of nonlinear spatially periodic solutions and, in particular, soliton and kink type solutions. Unfortunately such a method is not available to study the discrete form of equation. However, it is shown in section 2 that for particular forms for the potential $V(M)$ nonlinear solutions exist in the discrete case and these include kinks and solitons. Equations with such potentials are called integrable.

In the continuous case the stability analysis is based on the existence of a marginally stable mode. This existence arises directly from the spatial invariance of the governing equation (1.3). A perturbation method, appropriate to long wavelength disturbance, is then developed which gives in compact form a necessary condition for stability. The method was first used in plasma physics by Rowlands (1969) and then extensively developed by Infeld and Rowlands (1990).

The discrete versions of the underlying equations lack spatial invariance and the method used in the continuous case cannot be used to show the existence of a marginal mode. However, in the discrete case the time-independent solutions are arbitrary up to a phase factor and the requirement of spatial invariance can be replaced by one of phase invariance. The existence of a marginal mode is then easily shown. The rest of the perturbation theory follows quite closely that in the continuous case and a necessary condition for stability found. The method is discussed in detail in section 3.

2. Nonlinear time-independent (integrable) solutions

For both equations (1.1) and (1.2), the time independent solutions satisfy

$$\bar{M}_{n+1} + \bar{M}_{n-1} - 2\bar{M}_n = \tfrac{1}{2}\dot{V}(\bar{M}_n). \tag{2.1}$$

Unfortunately equations of this general type are known to have chaotic solutions, that is $\bar{M}_n$ varies chaotically with n. The best known example is where

$$\dot{V}(M) = 2K \sin M$$

in which case (2.1) is the discrete analogue of the simple pendulum, and is known as the standard map, and is ubiquitous in studies of Hamiltonian systems with two disparate frequencies (Chirikov 1979).

In design studies of particle accelerators McMillan (1971) considered equations of the form (2.1) and in particular the form for $V(M)$ which leads to bounded periodic and non-chaotic solutions. It is this type of solution we are interested in. Following McMillan we rewrite (2.1) in the form ($\bar{M} \to x$)

$$y_{n+1} = x_n \qquad x_{n+1} = -y_n + f(x_n) \tag{2.2}$$

that is $x_{n+1} + x_{n-1} = f(x_n)$ where $f(x) = 2x + \dot{V}(x)/2$. Now we demand that $x_{n+1} = g(x_n)$, where g is some function not necessarily single valued. Then $y_n = g^{-1}(x_n)$ and the condition for acceptable solutions is that $f(x)$ is of the form

$$f(x) = g(x) + g^{-1}(x). \tag{2.3}$$

This is not an easy equation to solve but McMillan did obtain some solutions, of which one is discussed further below. Guided by the method of phase-plane analysis, as applied to the continuous case, we impose the condition that equation (2.2) has solutions of the form

$$(x_{n+1} - x_{n-1})^2 = U(x_n) \tag{2.4}$$

which is equivalent to demanding that

$$y(x) = \left(f(x) \pm \sqrt{U(x)}\right)/2 \tag{2.5}$$

or

$$g(x) = \left(f(x) \mp \sqrt{U(x)}\right)/2.$$

McMillan obtained a solution to (2.3) by insisting that x, y satisfy the equation of the general conic section, the simplest non-trivial equation symmetric under the interchange of x and y, namely

$$Ax^2y^2 + B(x^2y + xy^2) + C(x^2 + y^2) + Dxy = H \tag{2.6}$$

where A, B, C, D and H are constants. This may be solved for y as a function x and compared to (2.5) to give

$$f(x) = -x(D + Bx)/(Ax^2 + Bx + C) \tag{2.7}$$

and

$$U(x) = \frac{x^2(Bx + D)^2 - 4(Cx^2 - H)(Ax^2 + Bx + C)}{(Ax^2 + Bx + C)^2}. \tag{2.8}$$

Thus we have shown the existence of a class of bounded periodic solutions of (2.1) corresponding to

$$\dot{V}(M) = 2f(M) - 4M \tag{2.9}$$

with $f(M)$ given by (2.7). Furthermore, such solutions also satisfy the equivalent of (2.4), namely

$$(\bar{M}_{n+1} - \bar{M}_{n-1})^2 = U(\bar{M}_n) \tag{2.10}$$

where $U(\bar{M}_n)$ is given by (2.8).

By analogy with phase-plane analysis for a continuous second-order differential equation, we see that if $U(M) \geqslant 0$ between two distinct values M_a and M_b then equation (2.10) corresponds to a closed curve in the discrete phase plane where $\bar{M}_{n+1} - \bar{M}_{n-1}$ is plotted as a function of $\bar{M}_n$. This curve corresponds to a solution of (2.1) of the form of a periodic or quasi-periodic variation of M_n with n. The imposition of the condition that $\dot{U}(M)$ is zero at M_a leads to a soliton solution whilst a kink corresponds to $\dot{U}(M) = 0$ at $M = M_a$ and M_b.

Quispel *et al* (1989) have discussed equations of the form

$$\bar{M}_{n+1} = \frac{f_1(\bar{M}_n) - \bar{M}_{n-1} f_2(\bar{M}_n)}{f_2(M_n) - M_{n-1} f_3(M_n)}$$

where $f_n(x)$ are products of two quadratic functions. They show that such maps satisfy an equation of the form (2.6) but with an added term $E(x+y)$, where $x \equiv \bar{M}_n$ and $y \equiv \bar{M}_{n+1}$. Importantly, the above equation can be written in the form of (2.10) with the value of U calculated from the above-mentioned extension of (2.6). Thus, as found by Quispel *et al* (1989), an equation of the above form can have solutions corresponding to closed curves in discrete phase plane and such solutions they call integrable.

Unlike the continuous case it is not possible to proceed from (2.10) to obtain an explicit solution for $\bar{M}_n$ as a function of n. Rather we cheat, assume that the n variation of $\bar{M}_n$ is that of an elliptic function and substitute into the basic equation (2.1). This procedure produces two distinct classes of solution in the form of nonlinear waves, one of which reduces to a soliton and the other to a kink in a suitable limit.

With $A = C = 1$, $B = 2$ and $D = 2q$ we have the basic equation

$$\bar{M}_{n+1} + \bar{M}_{n-1} + 2\bar{M}_n(q + \bar{M}_n)/(1 + \bar{M}_n)^2 = 0 \tag{2.11}$$

whose solution can be written in the form

$$\bar{M}_n = a - b\,\mathrm{sn}^2(\beta n + s, k) \tag{2.12}$$

where $\mathrm{sn}(x, k)$ is the Jacobi elliptic function and the constants a, b and β satisfy

$$(a+1)k^2\,\mathrm{sn}^2(\beta, k) = b$$

$$(a+1)^2\{1 - 2a\,\mathrm{cn}^2(\beta, k)\,\mathrm{dn}^2(\beta, k) - k^2\,\mathrm{sn}^4(\beta, k)\} = 1$$

and

$$(a+1)^2\{\mathrm{cn}^2(\beta, k) + \mathrm{dn}^2(\beta, k) + \mathrm{cn}^2(\beta, k)\,\mathrm{dn}^2(\beta, k)\} = 2 - q.$$

Here k is a parameter ($0 \leqslant k \leqslant 1$) which controls the amplitude and the wavelength of the nonlinear oscillations of $\bar{M}_n$ as a function of n, and s an arbitrary phase factor. For $k = 1$ we have a soliton solution of the form

$$\bar{M}_n = \sinh^2\beta\,\mathrm{sech}^2(n\beta + s) \tag{2.13}$$

with $\cosh(2\beta) = -q$.

For the above solutions the potential U has the form

$$U(M) = 4[M^2(q+M)^2 - (1+M)^2(M^2 - H)]/(1+M)^4. \tag{2.14}$$

H is a function of k with $H = 0$ corresponding to the soliton solution, that is $k = 1$.

For $A = -1$, $B = 0$, $C = 1$ and $D = -\alpha$ the basic equation becomes

$$\bar{M}_{n+1} + \bar{M}_{n-1} - \alpha\bar{M}_n/(1 - \bar{M}_n^2) = 0 \tag{2.15}$$

whose solution is

$$\bar{M}_n = k\,\mathrm{sn}(\beta, k)\,\mathrm{sn}(n\beta + s, k) \tag{2.16}$$

where the phase s is still arbitrary, β is given by the condition

$$\alpha = 2\,\mathrm{cn}(\beta, k)\,\mathrm{dn}(\beta, k) \tag{2.17}$$

and $0 \leqslant k \leqslant 1$. For $k = 1$ we have the kink solution

$$\bar{M}_n = \tanh\beta \tanh(n\beta + s) \tag{2.18}$$

where $\alpha = 2\,\mathrm{sech}^2\beta$. For the above solutions the potential is given by

$$U(M) = \{\alpha^2 M^2 + 4(1 - M^2)(H - M^2)\}/(1 - M^2)^2 \tag{2.19}$$

$H = (1 - \alpha/2)^2$ corresponds to the kink solution.

Broomhead and Rowlands (1983) based a perturbation theory on this particular case to discuss the onset of chaos in the standard map referred to above.

In summary we have shown that an equation such as (2.1), can have, by suitable choice of $V(M)$, bounded periodic solutions. Further, for two distinct choices, we have explicit solutions of $\bar{M}_n$ as a function of n. All these solutions may be written in the form

$$\bar{M}_n = h(n/T(k) + \alpha(k)s, k) \tag{2.20}$$

where $h(x, k)$ is periodic in x with period T and the phase parameter s is arbitrary. The parameter k controls the amplitude and period of the nonlinear solution. Equations (2.12) and (2.16) furnish specific examples which are such as to reduce to a soliton and kink solution, respectively, in the limit as $T \to \infty$. It is sometimes convenient to express (2.20) as a Fourier series and write

$$\bar{M}_n = \sum_{m=0}^{\infty} A_m(k) \cos\left(\left(\frac{n}{T(k)} + \alpha(k)s\right) 2\pi m\right). \tag{2.21}$$

It may be noted that though these solutions are continuous functions of s, if $T(k) = 2\pi p/q$, (where p and q are integers) then for any one value of s, $\bar{M}_n$ takes on only q distinct values. In a phase plane where $\bar{M}_{n+1} - \bar{M}_{n-1}$ is plotted as a function of $\bar{M}_n$, this would give q distinct points, whereas if $T(k)/2\pi$ is irrational, one has a continuous curve as for the solution of a differential equation.

The solutions discussed above have much in common with those of the Toda lattice equation (Toda 1981) which we write in the form

$$-\frac{m}{a}\frac{\mathrm{d}^2 r_n}{\mathrm{d}t^2} = (\psi_{n+1} + \psi_{n-1} - 2\psi_n) \tag{2.22}$$

where $\psi_n = (\exp(-br_n) - 1)$. Time periodic solutions exist of the form

$$\psi_n = \frac{\mathrm{sn}(2K/\lambda)}{d}\left\{\mathrm{dn}^2\left(2\left(\frac{n}{\lambda} + \nu t\right)K, k\right) - E/K\right\}$$

where E, K are the elliptic integrals of argument k, and

$$d = \mathrm{cn}^2(2K/\lambda) + (E/K)\,\mathrm{sn}^2(2K/\lambda)$$

where ν is a function of λ which itself is arbitrary. Interestingly, if we define

$$\phi_n = d\mathrm{e}^{-br_n}$$

then

$$(\phi_{n+1} - \phi_{n-1})^2 = U(\phi_n) \tag{2.23}$$

where

$$U(\phi) = A(1 - \phi)(\phi - \mathrm{cn}^2(2K/\lambda))(\phi - \mathrm{dn}^2(2K/\lambda))/\phi^4$$

and $A = 16\,\mathrm{cn}^2(2K/\lambda)\,\mathrm{dn}^2(2K/\lambda)$. The above expression is of the form (2.10): a form which automatically means that the solutions lie on simple curves in the discrete analogue of a phase plane where $\phi_{n+1} - \phi_{n-1}$ is plotted as a function of ϕ_n. In particular, chaotic behaviour is ruled out.

3. Linear stability analysis

We write the solution of (1.1) or (1.2) in the form

$$M_n(t) = \bar{M}_n + \delta M_n \mathrm{e}^{-\lambda t}$$

substitute into these equations and neglect products of δM_n to give the linearized eigenvalue equation

$$L\,\delta M_n = -\mu\,\delta M_n. \tag{3.1}$$

Here

$$L\,\delta M_n \equiv \delta M_{n+1} + \delta M_{n-1} - \left(2 + \frac{\ddot{V}}{2}(\bar{M}_n)\delta M_n\right) \tag{3.2}$$

and $\mu = \lambda/\gamma$ or $a\lambda^2$, respectively.

In the continuous case, where L is replaced by a second-order differential equation, Floquet's theorem can be invoked. By analogy, we write $\delta M_n = \psi_n \mathrm{e}^{\mathrm{i}ln}$ and take ψ_n to be periodic with the same period as $\bar{M}_n$. This gives

$$L\psi_n = -\mu\psi_n + (1 - \exp(\mathrm{i}l))\psi_{n+1} + (1 - \exp(\mathrm{i}l))\psi_{n-1} \tag{3.3}$$

which we solve by treating l as small and expanding in powers of l. Thus we write

$$\psi_n = \psi_n^{(0)} + l\psi_n^{(1)} + l^2\psi_n^{(2)} + \cdots$$

and

$$\mu = 0 + l\mu_1 + l^2\mu_2 + \cdots$$

and substitute into (3.3) to give

$$L\psi_n^{(0)} = 0 \tag{3.4}$$

$$L\psi_n^{(1)} = -\mu_1\psi_n^{(0)} - \mathrm{i}\left(\psi_{n+1}^{(0)} - \psi_{n-1}^{(0)}\right) \tag{3.5}$$

$$L\psi_n^{(2)} = -\mu_1\psi_n^{(1)} - \mu_2\psi_n^{(0)} - \mathrm{i}\left(\psi_{n+1}^{(1)} - \psi_{n-1}^{(1)}\right) + \tfrac{1}{2}\left(\psi_{n+1}^{(0)} + \psi_{n-1}^{(0)}\right) \tag{3.6}$$

plus higher order equations.

In the continuous case the solution of (3.4) was obtained by invoking the spatial invariance of (1.3) and was simply the spatial derivative of $\bar{M}$. Here we invoke the arbitrariness of the solution $\bar{M}_n$ with respect to the phase factors and obtain the result that the solution of (3.4) is

$$\psi_n^{(0)} = \partial\bar{M}_n/\partial s. \tag{3.7}$$

This is readily seen to be correct by differentiating equation (2.1) with respect to s. Thus the phase invariance implies the existence of a marginally stable mode just as the spatial invariance implied the existence of a similar mode in the continuous case.

The rest of the analysis follows very closely that for the continuous case. Thus we multiply (3.5) by $\psi_n^{(0)}$ and sum over n from 0 to $N-1$ where N corresponds to the period of ψ_n, that is $\psi_{n+N} = \psi_n$. (Note that N depends on k.) Using the result that

$$\sum_{n=0}^{N-1} \psi_n^{(0)} \left(\psi_{n+1}^{(1)} + \psi_{n-1}^{(1)}\right) = \sum_{1}^{N} \psi_n^{(1)} \psi_{n-1}^{(0)} + \sum_{-1}^{N-2} \psi_n^{(1)} \psi_{n+1}^{(0)}$$

$$= \sum_{0}^{N-1} \psi_n^{(1)} \left(\psi_{n+1}^{(0)} + \psi_{n-1}^{(0)}\right) + \left(\psi_N^{(1)} \psi_{N-1}^{(0)} - \psi_0^{(1)} \psi_{-1}^{(0)}\right)$$

$$+ \left(-\psi_{N-2}^{(1)} \psi_{N-1}^{(0)} + \psi_{-1}^{(1)} \psi_0^{(0)}\right)$$

$$= \sum_{0}^{N-1} \psi_n^{(1)} \left(\psi_{n+1}^{(0)} + \psi_{n-1}^{(0)}\right)$$

where we have used the periodicity of both $\psi_n^{(0)}$ and $\psi_n^{(1)}$, shows that the operator L is self-adjoint, in which case we obtain the consistency condition

$$\mu_1 \sum_{n=0}^{N-1} \left(\psi_n^{(0)}\right)^2 = -\mathrm{i} \sum_{n=0}^{N-1} \psi_n^{(0)} \left(\psi_{n+1}^{(0)} - \psi_{n-1}^{(0)}\right).$$

By changing the suffices and using the periodicity it is readily shown that the right-hand side of the above equation is identically zero. Thus $\mu_1 \equiv 0$. By direct substitution it may then be shown that the particular integral of (3.5) is given by

$$\psi_n^{(1)} = -\mathrm{i}n \frac{\partial}{\partial s} \bar{M}_n.$$

To obtain the complete solution we must add the homogeneous solutions in such a way to make $\psi_n^{(1)}$ periodic. One homogeneous solution is given by (3.7) while the second is $\partial \bar{M}_n / \partial k$ where k is the parameter which labels the various static solutions. Using the form for $\bar{M}_n$ given by (2.21) we add to the above form for $\psi_n^{(1)}$ sufficient amount of the second homogeneous solution to remove the secular behaviour associated with the factor n. This gives a periodic solution for $\psi_n^{(1)}$ of the form

$$\psi_n^{(1)} = \mathrm{i}\alpha(k) \Big/ \left(\frac{\mathrm{d}}{\mathrm{d}k}(1/T(k))\right) \sum_{m=0}^{\infty} \left\{ \frac{\mathrm{d}A_m}{\mathrm{d}k} \cos(m\theta_n) - m A_m 2\pi s \frac{\mathrm{d}\alpha}{\mathrm{d}k} \sin(m\theta_n) \right\} \tag{3.8}$$

where $\theta_n = (n/T(k) + s\alpha(k))2\pi$, or in the form

$$\psi_n^{(1)} = -\frac{\mathrm{i}\alpha T^2}{\mathrm{d}T/\mathrm{d}k} \left\{ \left(\sum_{m=0}^{\infty} \frac{\mathrm{d}A_m}{\mathrm{d}k} \cos(m\theta_n) \right) + \frac{1}{\alpha} \frac{\mathrm{d}\alpha}{\mathrm{d}k} \frac{\partial \bar{M}_n}{\partial s} \right\}. \tag{3.9}$$

To next order in the expansion in l, the consistency condition gives an equation for μ_2, namely,

$$\mu_2 \sum_{n=0}^{N-1} \left(\frac{\partial \bar{M}_n}{\partial s}\right)^2 = -\mathrm{i} \sum_{n=0}^{N-1} \frac{\partial \bar{M}_n}{\partial s} \left(\psi_{n+1}^{(1)} - \psi_{n-1}^{(1)}\right) + \sum_{n=0}^{N-1} \frac{\partial \bar{M}_n}{\partial s} \frac{\partial \bar{M}_{n+1}}{\partial s}. \tag{3.10}$$

This expression must be independent of the phase so we integrate both sides with respect to s from 0 to $1/\alpha$. First consider the left-hand side of (3.10). Using the expansion as given by (2.21) allows us to write

$$\int_0^{1/\alpha} \mathrm{d}s \left(\frac{\partial \bar{M}_n}{\partial s}\right)^2 = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} 4\pi^2 ml\alpha^2 A_m A_l I_{m,l}$$

where

$$\begin{aligned} I_{m,l} &= \int_0^{1/\alpha} \mathrm{d}s \sin\left[\left(\frac{n}{T}+\alpha s\right)2\pi m\right]\sin\left[\left(\frac{n}{T}+\alpha s\right)2\pi l\right] \\ &= \frac{1}{2}\int_0^{1/\alpha} \mathrm{d}s \left\{\cos\left(\left(\frac{n}{T}+\alpha s\right)2\pi(m-l)\right) - \cos\left(\left(\frac{n}{T}+\alpha s\right)2\pi(m+l)\right)\right\} \\ &= \frac{1}{2\alpha}\delta_{m,l} \qquad m>0,\ l>0 \end{aligned}$$

so that

$$\int_0^{1/\alpha} \mathrm{d}s \left(\frac{\partial \bar{M}_n}{\partial s}\right)^2 = 2\pi^2\alpha \sum_{m=1}^{\infty} A_m^2 m^2.$$

Similarly

$$\int_0^{1/\alpha} \mathrm{d}s \frac{\partial \bar{M}_n}{\partial s}\frac{\partial \bar{M}_{n+1}}{\partial s} = 2\pi^2\alpha \sum_{m=1}^{\infty} A_m^2 m^2 \cos(2\pi m/T).$$

To calculate the remaining contribution we first note that by change of suffix in the second summation

$$\begin{aligned} \sum_{n=0}^{N-1} \frac{\partial \bar{M}_n}{\partial s}\left(\frac{\partial \bar{M}_{n+1}}{\partial s} - \frac{\partial \bar{M}_{n-1}}{\partial s}\right) &= \sum_{n=0}^{N-1} \frac{\partial \bar{M}_n}{\partial s}\frac{\partial \bar{M}_{n+1}}{\partial s} - \sum_{-1}^{N-2} \frac{\partial \bar{M}_{n+1}}{\partial s}\frac{\partial \bar{M}_n}{\partial s} \\ &= +\frac{\partial \bar{M}_{N-1}}{\partial s}\frac{\partial \bar{M}_N}{\partial s} - \frac{\partial \bar{M}_{-1}}{\partial s}\frac{\partial \bar{M}_0}{\partial s} \end{aligned}$$

which is zero because of the periodicity of $\bar{M}_n$ ($\bar{M}_N = \bar{M}_0$). We now use the expression (3.9) for $\psi_n^{(1)}$ and because of the above result we see that the second term in this expression will not contribute to the following integral and we can write

$$\begin{aligned} &\sum_{n=0}^{N-1} \int_0^{1/\alpha} \mathrm{d}s \frac{\partial \bar{M}_n}{\partial s}\left(\psi_{n+1}^{(1)} - \psi_{n-1}^{(1)}\right) \\ &\qquad = \frac{+\mathrm{i}\alpha T^2}{\mathrm{d}T/\mathrm{d}k}\sum_{n=0}^{N-1}\int_0^{1/\alpha} \mathrm{d}s\left(\sum_{m=0}^{\infty}\frac{\mathrm{d}A_m}{\mathrm{d}k}\cos m\theta_n\right)\left(\frac{\partial \bar{M}_{n+1}}{\partial s} - \frac{\partial \bar{M}_{n-1}}{\partial s}\right) \\ &\qquad = \frac{-\mathrm{i}T^2}{\mathrm{d}T/\mathrm{d}k} 2\pi N \sum_{m=1}^{\infty} \frac{\mathrm{d}Am}{\mathrm{d}k} A_m m \sin\left(\frac{2\pi m}{T}\right). \end{aligned}$$

This eventually gives an equation for μ_2 in the form

$$\mu_2 \sum_{m=1}^{\infty} m^2 A_m^2 = \frac{-T^2/\pi}{(\mathrm{d}T/\mathrm{d}k)} \sum_{m=1}^{\infty} A_m \frac{\mathrm{d}A_m}{\mathrm{d}k} m \sin\left(\frac{2\pi m}{T}\right) + \sum_{m=1}^{\infty} A_m^2 m^2 \cos(2\pi m/T).$$

If we define

$$G = \sum_{m=1}^{\infty} A_m^2 m \sin(2\pi m/T) \tag{3.11}$$

and

$$I^2 = \sum_{m=1}^{\infty} m^2 A_m^2 \tag{3.12}$$

we can write

$$\mu_2 = -\frac{\partial G/\partial k(T^2/I^2)}{2\pi(\mathrm{d}T/\mathrm{d}k)}. \tag{3.13}$$

It is readily shown that G, as defined by (3.11), may be written in the form

$$G = \frac{1}{2\pi} \int_0^{1/\alpha} \frac{\partial \bar{M}_n}{\partial s} (\bar{M}_{n+1} - \bar{M}_{n-1})_{\mathrm{d}s}.$$

It should be pointed out that this expression for μ_2 has been obtained using only the fact that $\bar{M}_n$ satisfies (2.1) and can be written in the form (2.21). It has not been necessary to use an explicit form for the solution such as (2.12) or (2.16).

Using the result given by (2.10) and changing the integration with respect to s by one with respect to M gives

$$G = \frac{1}{\pi} \int_{\bar{M}_a}^{\bar{M}_b} \mathrm{d}M \sqrt{U(M)}$$

where $\bar{M}_{b,a}$ are the two roots of $U(M) = 0$ defining a region where $U(M) > 0$. This expression is analogous to that for the quantity G as defined by equation (2.19) of GIRW but the k dependence of $U(M)$ is not as simple as in the continuous case and so no simple expression for $\partial G/\partial k$ is available. However, it is obviously positive and for the two examples given in section 2 we see that as we increase k from zero to 1, so that the solution goes from small amplitude waves to a kink or a soliton, the value of G increases. That is $\partial G/\partial k$ is positive. Then, in complete analogy with the continuous case discussed in GIRW, the sign of the quantity μ depends simply on that of $\mathrm{d}T/\mathrm{d}k$. Again this is positive in the cases under discussion so we may conclude that $\mu_2 < 0$. This means that, as for the continuous case, the nonlinear wave solutions described by (1.1) are unstable and by analogy with the continuous case the growth rates go to zero as $k \to 1$, that is, the kink or soliton solution.

If we apply the result to equation (1.2) it is seen that the nonlinear wave solutions are stable to the long wavelength disturbances considered here and equation (3.13) is now an expression for the frequency of oscillation.

In a second paper, Infeld *et al* (1991) showed that in the continuous case the soliton solution of the Landau–Ginsburg equation was unstable to a class of perturbations (not included in the class considered in GIRW or above), namely where δM_n is proportional to $a + bM_n^2$ with a and b depending on μ and with μ taking negative values. The kink was found to be stable. It is not been possible to find analytically the analogous class of perturbations for the discrete equations discussed in the present paper. However, by extending a well known result in the theory of linear second-order differential equations to difference equations it is possible to show that the discrete soliton solution is unstable but the discrete kink is stable. The details are given in the appendix.

4. Choice of potential

In most applications of the Landau–Ginsburg equation the actual form of the potential function of $V(M)$ as a function of M is either very complicated or not known in closed form. It is then usually approximated by a simple polynomial in M which takes account of any symmetry requirement on $V(M)$. Thus (2.15) would be approximated by

$$(\bar{M}_{n+1} + \bar{M}_{n-1} - 2\bar{M}_n) - \alpha\bar{M}_n(1 + \bar{M}_n^2) = 0. \tag{4.1}$$

The solution of this equation should be compared to that of (2.15). For small values of $\bar{M}_n$ the phase-plane contours are quite similar and show the existence of periodic nonlinear waves. However, for contours which approach the sepratrices there is a fundamental difference. Solutions of (4.1) show the breakup of closed contours and the advent of chaotic trajectories while solutions of (2.14) retain their simplicity right up to the kink solution which originates with the sepratrices.

Thus if it is thought that any physical quantity corresponding to the quantity $\bar{M}_n$ in equation (2.1) cannot be a chaotic function of n then this requirement places stringent conditions on the form of $V(M)$, namely the form implicit in equation (2.15) but not that in equation (4.1). Since chaotic behaviour is always more apparent near sepratrices, which physically correspond to soliton or kink behaviour, and such solutions are usually the more stable (as shown in section 3 in the discrete case and in GRIW for the continuous case) and hence more likely to exist in nature, it is the naturally occurring solutions which will be chaotic.

In solid-state physics and in biological studies of large protein molecules, for example, where the presence of an underlying lattice is very basic, the question of whether measurable quantities can be chaotic is of a fundamental nature. If such solutions are allowed then the nature of the potential $V(M)$ is not too critical but the continuous approximation cannot be made as it loses the very nature of the solution. On the other hand if chaotic solutions are not acceptable then the potential $V(M)$ must be of a special nature leading to an integrable solution. Passing to the continuous limit in this case causes no problems.

If one is to demand that the underlying discrete equations must be integrable at least in the time-independent case then it may be best to base the theory on an equation of the form of (2.10) rather than (2.1) as this ensures integrability with little restriction on the form of $U(M)$. In the continuous case the equation analogous to (2.10) and the one analogous to (2.1) are simply related by differentiation and with $U(M) = V(M) + H$ where H is some constant. Unfortunately this is not true in the discrete case and the relationship in general is not known.

5. Conclusions

It has been shown that for a class of potentials, time-independent solutions of the discrete Landau–Ginsburg equation exist which are non-chaotic. However all such solutions are unstable with the exception of that of the form of a discrete kink. This behaviour is analogous to that found earlier for the case of the continuous Landau–Ginsburg equation.

The method of stability analysis as discussed in section 3 is based on the existence of the solution given by (3.7). Thus the method should be extendable to more complicated sets of equations as in the analogous method in the continuous case (see for example Bridges and Rowlands 1994).

Appendix

We write the discrete eigenvalue equation (3.1) in the form

$$\delta M_{n+1} + \delta M_{n-1} + g_n \delta M_n = -\mu \delta M_n \tag{A.1}$$

and consider another possible solution $\delta \hat{M}_n$ corresponding to an eigenvalue $\hat{\mu}$ so that

$$\delta \hat{M}_{n+1} + \delta \hat{M}_{n-1} + g_n \delta \hat{M}_n = -\hat{\mu} \delta \hat{M}_n. \tag{A.2}$$

We multiply (A.1) by $\delta \hat{M}_n$, (A.2) by δM_n, subtract and sum over n from $-N$ to m to give

$$(\delta M_{m+1} \delta \hat{M}_m - \delta M_m \delta \hat{M}_{m+1}) - (\delta M_{-N} \delta \hat{M}_{-N-1} - \delta M_{-N-1} \delta \hat{M}_{-N}) = -(\mu - \hat{\mu}) \sum_{-N}^{m} \delta M_n \delta \hat{M}_n. \tag{A.3}$$

We now choose N such that the second bracketed term is zero. For perturbations to kink and soliton type solutions this corresponds to $N = \infty$, while for nonlinear waves we choose $\delta M_{-N} = \delta \hat{M}_{-N} = 0$.

Suppose now that δM_n is positive for $-N \leqslant n \leqslant m$ but $\delta M_{m+1} < 0$ while $\delta \hat{M}_n$ remains positive at least for $-N \leqslant n \leqslant m+1$, then first, the left-hand side of (A.3) is negative, and second, the summation on the right-hand side is positive. Under these circumstances we must have

$$\mu > \hat{\mu}. \tag{A.4}$$

This condition may be interpreted by saying that the eigenfunction with the least number of zeros, in this case $(\delta \hat{M}_n)$, has the smallest value of μ.

Now in the context of the present paper we have shown there always exists an eigenfunction proportional to $\partial M_n / \partial s$ with eigenvalue $\mu = 0$. For a soliton solution, apart from the zeros at infinity which are not relevant as they are accounted for by the choice of N, $\partial M_n / \partial s$ has one zero and hence by the above result there exists an eigenvalue $\delta \hat{M}_n$ with an eigenvalue $\hat{\mu} < 0$. This corresponds to an instability and hence the soliton solution is linearly unstable. For the kink solution, the eigenfunction $\partial M_n / \partial s$ corresponding to $\mu = 0$ is always of one sign and hence no eigenfunction with a lower value of μ exists. Thus the kink mode is marginally stable.

The above method also shows that an unstable eigenfunction exists in the case of nonlinear waves confirming the stability analysis given in section 3.

The analysis leading to the result (A.4) is a straightforward extension of that used in the continuous case to prove a similar result (see for example Morse and Feshbach 1953, section 6.3).

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